JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **203,** 333–351 (1996)
ARTICLE NO. 0383

One Pointed Univalent Logharmonic Mappings

Zayid Abdulhadi*

Department of Mathematical Sciences, KFUPM, Dhahran, Saudi Arabia

and

Département des Mathématiques et Statistiques, Université Laval, Québec, G1K 7P4 Canada

Submitted by H. M. Srivastava

Received July 22, 1994

A complete characterization of univalent logharmonic mappings from the exterior of the unit disk U, Δ onto $\mathbb{C} \setminus \{1\}$ is given.

1. INTRODUCTION

Let Ω be a domain in the complex plane $\mathbb{C}$. We shall say that Ω is harmonically accessible from a domain D if there exists a univalent harmonic mapping from D onto Ω. If, in addition, D is also harmonically accessible from Ω then we say that D and Ω are harmonically equivalent. In particular, conformally equivalent domains are also harmonically equivalent. There are domains D and Ω where Ω is harmonically accessible from D but they are not harmonically equivalent. For example, the pointed unit disk $\Omega = U \setminus \{0\}$ is harmonically accessible from the annulus $D = \{z;\ r < |z| < R\}$, $0 < r < R$ but not vice versa. More generally, given any doubly connected domain D of $\mathbf{C}$, then there exists a univalent harmonic mapping F from D onto Ω where Ω is either the pointed unit

* This work was supported in part by King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia. E-mail: facl004@saupm00.bitnet.

† Supported by grants from the NSERC (Canada) and FCAR (Quebec). E-mail: walheng@mat.ulaval.ca.

0022-247X/96 $18.00

disk or the pointed plane $\mathbb{C} \setminus \{0\}$. (For more details see, e.g., [4, 8, 11].) It follows that, in general, one cannot simply choose $D = \Omega$ in order to characterize a nonparametric minimal surface by a univalent harmonic map F from D onto Ω. If Ω is harmonically accessible from D and if D is conformally equivalent to G, then Ω is harmonically accessible from G. On the other hand, since composition of harmonic mappings are, in general, not harmonic the property of harmonically accessible is not transitive. Several authors also have studied harmonic mappings between Riemannian manifolds and excellent surveys have been given, e.g., in [5, 6, 9, 10, 13, 14].

Although any domain of $\mathbb{C}$ containing the point infinity can be mapped onto a pointed plane $\Omega = \mathbb{C} \setminus \bigcup_{j \in \mathbf{J}} \{w_{\mathbf{j}}\}$, $\mathbf{J} \subset \mathbf{N}$, by a univalent harmonic mapping, the points w_j cannot be prescribed in general. However, one can do it in the following case.

Let K be a compact continuum of $\mathbb{C}$ which satisfies the following two conditions:

(a) $\mathbb{C} \setminus K$ is connected,

(b) $z \in K$ implies that $z + 2\pi in \notin K \ \forall n \in \mathscr{Z} \setminus \{0\}$.

Define

$$K_n = \{z = \zeta + 2\pi in;\ \zeta \in K\}, \qquad n \in \mathscr{Z},$$

$$D^* = \mathbb{C} \setminus \bigcup_{n \in \mathscr{Z}} K_n \tag{1}$$

and

$$\Omega^* = \mathbb{C} \setminus \bigcup_{n \in \mathscr{Z}} \{2\pi in\}. \tag{2}$$

The purpose of this paper is to show that Ω^* is harmonically accessible from D^*. In other words, we show explicitly the existence and the behavior of univalent orientation-preserving harmonic mappings F from D^* onto Ω^*. In order to achieve our goal, we consider instead of F the logharmonic mapping $f(z) = \exp F(\log z)$, which is defined on the domain $D_1^* = \mathbb{C} \setminus \exp(K)$. Applying a conformal mapping from D_1^* onto the complement of the closed unit disk $\overline{U}$, $\Delta = \mathbb{C} \setminus \overline{U}$, we are led to study univalent logharmonic mappings defined on Δ. Such mappings will be introduced in Section 2. We show in (Theorem 2.1) that they are of the form

$$f(z) = z|z|^{2\beta}\left(\frac{z-p}{1-\bar{p}z}\right)\left|\frac{z-p}{1-\bar{p}z}\right|^{2\gamma} h\bar{g}, \tag{3}$$

where $\operatorname{Re}\beta > -1/2$, $\operatorname{Re}\gamma > -1/2$, $|p| \geq 1$, and where h and g are analytic nonvanishing functions on Δ satisfying $h(\infty) \in \mathbb{C} \setminus \{0\}$ and $g(\infty) = 1$. We always take for $|z|^{2\beta}(|(z-p)/(1-pz)|^{2\gamma}$ resp.) the branch defined by $1^{2\beta} = 1^{2\gamma} = 1$. If f is nonvanishing on Δ, we may choose $p = 1$. Univalence criteria are given in Theorem 2.2 and Theorem 2.3. We may multiply f by a positive constant and also apply a rotation to the preimage. In other words, we may assume that $p \geq 1$ and that $|h(\infty)| = 1$. The family of these univalent logharmonic exterior mappings on Δ will be denoted by Σ_{Lh} (Section 3). Mappings belonging to this class have the property that $\partial U \cap f(\partial U) \neq \emptyset$ where ∂U denotes the unit circle $\{z;\ |z| = 1\}$ (Theorem 3.2).

The basic result of this article is Theorem 4.4 where we give a complete characterization of the class of univalent logharmonic mappings from Δ onto $\mathbb{C} \setminus \{1\}$ which makes it possible to show (Theorem 5.1) that Ω^* is harmonically accessible from the domain D^*. Finally, we give an elementary application to nonparametric minimal surfaces over Ω^*.

We may express Theorem 5.1 in geometric terms as follows. Consider a univalent harmonic mapping from a surface into the flat cylinder $S^1 \times \mathbb{R}$. If we take the conformal model for the flat cylinder, then it is the punctured plane $\mathbb{C} \setminus 0$. A logharmonic mapping, which we shall define in the next section, is a harmonic mapping into $\mathbb{C} \setminus 0$ endowed with the flat metric $|dw|^2/|w|^2$. Theorem 5.1 states then that the cylinder with a point removed is harmonically accessible from the cylinder with any compact continuum removed, provided that it remains a connected set. We are also led to the more general question as to whether the cylinder can be replaced by any Riemann surface with its complete metric of constant curvature.

2. UNIVALENT LOGHARMONIC EXTERIOR MAPPINGS

Let D be a domain of the complex plane $\mathbb{C}$. We say that a function f defined on D is a logharmonic mapping if f is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = a \frac{\bar{f}}{f} f_z, \tag{4}$$

where $|a(z)| < 1$ on D and where a belongs to $H(D)$.

Suppose that f is a nonconstant logharmonic mapping defined on D and fix a point $z_0 \in D$. By Lemma 2.2 in [1], there is a nonnegative integer

m and a complex number γ, $\operatorname{Re}\gamma > -1/2$ such that f admits in a certain neighborhood $V(z_0)$ of z_0 the following representation:

$$f(z) = (z - z_0)^m |z - z_0|^{2m\gamma} h_{z_0}(z)\overline{g_{z_0}(z)}. \tag{5}$$

The functions h_{z_0} and g_{z_0} are analytic and nonvanishing on $V(z_0)$. Note that $f(z_0) \neq 0$ if and only if $m = 0$ and that a univalent logharmonic mapping vanishes at z_0 if and only if $m = 1$. Univalent logharmonic mappings defined on the unit disk U have been studied in detail (in [1, 2]).

In this section we consider univalent logharmonic and orientation-preserving mappings f defined on the exterior of the unit disk U, $\Delta = \{z; |z| > 1\}$, satisfying $f(\infty) = \infty$. These mappings are called univalent logharmonic exterior mappings. If f does not vanish on Δ, then $\Psi(z) = 1/f(1/z)$ is a univalent logharmonic mapping defined on U normalized by $\Psi(0) = 0$. Moreover, $F(\zeta) = \log f(e^\zeta)$ is a univalent harmonic mapping defined on the right half-plane $\{\zeta;\ \operatorname{Re}\zeta > 0\}$ satisfying the relation $F(\zeta + 2\pi i) = F(\zeta) + 2\pi i$ and F is a solution of the linear elliptic partial differential equation,

$$\overline{F_{\bar\zeta}} = AF_\zeta ,$$

where the second dilatation function $A(\zeta) = a(e^\zeta)$ satisfies $A(\zeta + 2\pi i) = A(\zeta)$ on $\{\zeta;\ \operatorname{Re}\zeta > 0\}$. Such mappings were studied in [3, 7].

Suppose now that f vanishes on Δ. We shall show in Theorem 2.1 that f has to be of the form (3),

$$f(z) = z|z|^{2\beta}\left(\frac{z - p}{1 - \bar{p}z}\right)\left|\frac{z - p}{1 - \bar{p}z}\right|^{2\gamma} h\bar{g},$$

where $\operatorname{Re}\beta > -1/2$, $\operatorname{Re}\gamma > -1/2$, and where h and g are nonvanishing analytic functions on $\Delta \cup \{\infty\}$. Observe that not each function of the form (3) is univalent. Indeed, the example given in (31) of this paper,

$$f(z) = \bar{z}|z|^2 \frac{z - 4}{1 - 4\bar{z}},$$

is not a univalent logharmonic mapping on Δ but it can be written in the form (3) by putting $\beta = 1$, $\gamma = 0$, $p = 4$, $h(z) = 1/g(z) = (4z - 1)/4z$.

Again, we may connect f with a univalent harmonic mapping F in the following way. Define

$$\phi(z) = \frac{(z - p)(1 - \bar{p}z)}{z}.$$

Then

$$F(\zeta) = \log f\left(\phi^{-1}(e^\zeta)\right)$$

is a univalent harmonic mapping defined on the domain D^* (see (1)) where K is the horizontal interval $[2\ln(|p|-1), 2\ln(|p|+1)]$.

It is also interesting to note that the nonvanishing univalent logharmonic mappings f can also be expressed by the form (3) by putting $p=1$. In other words, there is no need to distinguish each time if f vanishes at a point $p \in \Delta$ or if f is nonvanishing on Δ. Suppose now that f is univalent and logharmonic in a neighborhood $V(\infty)$ of infinity and that $f(\infty)=\infty$. Without loss of generality, we may assume that f does not vanish on $V(\infty)$ and that $V(\infty) \cup \{\infty\}$ is a simply connected domain of the Riemann sphere. Applying relation (5) to the univalent logharmonic mapping $F(z) = 1/f(1/z)$, we get the representation

$$f(z) = z|z|^{2\beta} h_\infty(z)\overline{g_\infty(z)}, \qquad z \in V(\infty). \tag{6}$$

Our first result is a global representation of univalent logharmonic exterior mappings.

Theorem 2.1. *Let f be a univalent logharmonic mapping defined on the exterior Δ of the closed unit disk $\overline{U}$ such that $f(\infty)=\infty$. Suppose that $f(p)=0$ for some $p \in \Delta$ or if f does not vanish, let $p=1$. Then there are two complex numbers β and γ, $\operatorname{Re}\beta > -1/2$, $\operatorname{Re}\gamma > -1/2$, and two nonvanishing analytic functions h and g on $\Delta \cup \{\infty\}$ such that $g(\infty)=1$ and (3) holds; i.e., we have*

$$f(z) = z|z|^{2\beta}\left(\frac{z-p}{1-\bar{p}z}\right)\left|\frac{z-p}{1-\bar{p}z}\right|^{2\gamma} h\bar{g}$$

for all $z \in \Delta$.

Remark. If f is nonvanishing on Δ then (by putting $p=1$) the relation (3) is of the form (6).

Proof. (a) If f is nonvanishing on Δ then $F(z) = 1/f(1/z)$ is univalent on U and Theorem 2.1 follows from Lemma 2.1 in [1] and the above remark.

(b) Suppose $f(p)=0$ for some $p \in \Delta$. Then, by (5), f can be represented in a neighborhood $V(p)$ of p by

$$f(z) = (z-p)|z-p|^{2\gamma} h_p(z)\overline{g_p(z)},$$

or, equivalently, by

$$f(z) = \left(\frac{z-p}{1-\bar{p}z}\right)\left|\frac{z-p}{1-\bar{p}z}\right|^{2\gamma} H_p(z)\overline{G_p(z)}.$$

Furthermore, we have

$$f(z) = z|z|^{2\beta} h_\infty(z)\overline{g_\infty(z)}$$

in a neighborhood of infinity. Define

$$\Psi(z) = \frac{f(z)}{z|z|^{2\beta}((z-p)/(1-\bar{p}z))|(z-p)/(1-\bar{p}z)|^{2\gamma}}, \qquad z \in \Delta \quad (7)$$

and

$$\omega(\zeta) = \Psi\left(\frac{1}{\zeta}\right), \qquad \zeta \in U. \quad (8)$$

Then ω is locally the product of a nonvanishing analytic function and a nonvanishing anti-analytic function defined on U. Suppose that on an open connected set O of U, we have

$$\omega(\zeta) = H_1(\zeta)\overline{G_1(\zeta)} = H_2(\zeta)\overline{G_2(\zeta)}, \qquad \zeta \in O.$$

Then it follows that $H_1(\zeta)/H_2(\zeta)$ and $G_1(\zeta)/G_2(\zeta)$ are constant on O. Since U is simply connected we conclude from the Monodromy theorem that there are two nonvanishing analytic functions $H(\zeta)$ and $G(\zeta)$ on U such that

$$\omega(\zeta) = H(\zeta)\overline{G(\zeta)}$$

and Theorem 2.1 follows. ∎

Our next result gives a univalence criterion.

THEOREM 2.2. *Let f be a mapping of the form* (3). *Assume that the boundary values $f(e^{it})$ exist, $f(e^{it}) \neq 0$, for all $t \in [0, 2\pi)$ and that $f(e^{it})$ moves continuously on a Jordan curve Γ once around in the positive sense. Then f is a univalent logharmonic mapping defined on Δ if and only if $|a(z)| < 1$ for all $z \in \Delta$.*

Remark. We do not require that $f(e^{it})$ moves monotonically.

Proof. (a) Assume that there is a $z_0 \in \Delta$ such that $|a(z_0)| > 1$. Since $\operatorname{Re} \beta > -1/2$, we have $|a(\infty)| = |\overline{\beta}/(1+\beta)| < 1$ and therefore, there exists a $z_1 \in \Delta$ such that $|a(z_1)| = 1$, which implies that f is at least two valent in any neighborhood of z_1 (see, e.g., [3]). Hence, f is not univalent on Δ.

(b) Suppose that $|a| < 1$ on Δ. Since $a \in H(\Delta)$, the point infinity is a removable singularity of a. From the form (3), we conclude that there is an $R_0 > 1$ such that f is univalent on each circle $\{z; |z| = R\}$, $R \geq R_0$, and that its images are simple closed $\mathbb{C}^\infty$-*curves*. Choose a $w_0 \in f(\Delta)$. Then there is an $R_1 \geq R_0$ such that

$$I_R = \frac{1}{2\pi}\oint_{|z|=R} d\arg\big(f(z) - w_0\big) = 1 \qquad \text{for all } R > R_1$$

and there is an $r > 1$, close to 1, such that

$$I_r = \frac{1}{2\pi}\oint_{|z|=r} d\arg\big(f(z) - w_0\big) \geq 0.$$

The later statement follows from the facts that $f(e^{it})$ is continuous on ∂U and that $\log \omega(\zeta)$, defined in (8), is continuous on $\overline{U}$ and harmonic on U. Therefore, we have

$$0 \leq \frac{1}{2\pi}\oint_{|z|=R} d\arg\big(f(z) - w_0\big) - \frac{1}{2\pi}\oint_{|z|=r} d\arg\big(f(z) - w_0\big) \leq 1. \quad (9)$$

Since $w_0 \in f(\Delta)$, we conclude that there is exactly one $z_0 \in \{z; r < |z| < R\}$ which is mapped onto w_0. This holds for all $R \geq R_0$ and all $r > 1$ close to 1 and the univalence of f follows. ∎

The same idea of the proof can be applied when $f(\partial U)$ lies on a (nonclosed) Jordan arc Γ.

THEOREM 2.3. *Let f be a mapping of the form* (3). *Assume that the cluster sets* $\{f(e^{it}); 0 \leq t < 2\pi\}$ *lie on a free compact Jordan arc* Γ. *Then f is univalent if and only if* $|a| < 1$ *on* Δ.

Proof. The proof of the necessity of $|a| < 1$ uses the same arguments as given in the proof of the previous theorem. Suppose now that $|a| < 1$ on Δ. Choose $w_0 \notin \Gamma$. Then for $r > 1$, sufficiently close to 1 we have $(1/2\pi)\oint_{|z|=r} d\arg(f - w_0) = 0$. Together with (9) we conclude that there is exactly one $z_0 \in \{z; r < |z| < R\}$, $R \geq R_0$, which is mapped onto w_0 where R_0 is defined as in the proof of Theorem 2.2. This holds for all $R > R_0$ and $r > 1$ close enough to 1. Hence, w_0 belongs to the image $f(\Delta)$ and its preimage is a single point in Δ which shows that f is univalent on Δ. ∎

COROLLARY 2.4. *Let f be a mapping of the form* (3) *and assume that the boundary values* $f(e^{it})$ *exist for all* $t \in [0, 2\pi)$ *and that* $f(e^{it}) \equiv q$ *for some* $q \in \mathbb{C} \setminus \{0\}$. *Then f is a univalent logharmonic mapping from* Δ *onto* $\mathbb{C} \setminus \{q\}$ *if and only if* $|a| < 1$ *on* Δ.

Remark. The condition that the boundary values $f(e^{it})$ exist everywhere is important as the following example shows: Consider

$$f(z) = \bar{z}\frac{(z-1)}{(\bar{z}-1)} = z\frac{(1-1/z)}{(1-1/z)}.$$

Then $f(e^{it}) \equiv -1$ for all $t \in (0, 2\pi)$. But the set of cluster points at $z = 1$ is the whole unit circle and we have $f(\Delta) = \Delta$.

The next section deals with normalized univalent logharmonic exterior mappings.

3. THE CLASSES Σ_{Lh} AND Σ'_{Lh}

Let f be a univalent logharmonic exterior mapping defined on the unit disk U. Applying an appropriate rotation to the preimage, we may assume that $p \geq 1$. Furthermore, multiplying f by a positive constant we may assume the relation (3) holds with $g(\infty) = 1$ and $|h(\infty)| = 1$. Hence, denote by Σ_{Lh} the set of all univalent logharmonic mappings defined on Δ which are of the form

$$f(z) = z|z|^{2\beta}\left(\frac{z-p}{1-pz}\right)\left|\frac{z-p}{1-pz}\right|^{2\gamma} h\bar{g},$$

where $p \geq 1$, $\operatorname{Re}\beta > -1/2$, $\operatorname{Re}\gamma > -1/2$, and where h and g are analytic nonvanishing functions on $\Delta \cup \{\infty\}$ normalized by $g(\infty) = 1$ and $|h(\infty)| = 1$.

Fix τ, $\operatorname{Re}\tau > -1/2$. Then the transformation

$$T_f = f|f|^{2\tau} \tag{10}$$

is an isomorphism from Σ_{Lh} onto Σ_{Lh}. In particular, we have

$$\begin{aligned} T(f(z)) &= T\left(z|z|^{2\beta}\left(\frac{z-p}{1-pz}\right)\left|\frac{z-p}{1-pz}\right|^{2\gamma} h\bar{g}\right) \\ &= z|z|^{2\tilde{\beta}}\left(\frac{z-p}{1-pz}\right)\left|\frac{z-p}{1-pz}\right|^{2\tilde{\gamma}} H\overline{G}, \end{aligned}$$

where

$$\tilde{\beta} = \beta + \tau(1 + 2\operatorname{Re}\beta) \qquad \text{and} \qquad \tilde{\gamma} = \gamma + \tau(1 + 2\operatorname{Re}\gamma).$$

Observe that $\operatorname{Re}\tilde{\beta} > -1/2$ and $\operatorname{Re}\tilde{\gamma} > -1/2$. Therefore, we may restrict our attention to the subclass Σ'_{Lh} of Σ_{Lh} defined by

$$\Sigma'_{Lh} = \{f \in \Sigma_{Lh};\, \beta = 0\}. \tag{11}$$

Let $f \in \Sigma'_{Lh}$. Then $f(z) = (-z/p^{1+2\gamma})(h(\infty) + o(1))$ as $z \to \infty$ which shows that Σ'_{Lh} is not compact with respect to the topology of the locally uniform convergence. But if we fix p, $p > 1$, then we get the following result:

THEOREM 3.1. *Let $p_0 > 1$ be given. Then the set*

$$\mathscr{F}_{p_0} = \{f \in \Sigma'_{Lh},\, p = p_0\}$$

is compact with respect to the topology of the locally uniform convergence.

Proof. Since $\beta = 0$ we have $a(\infty) = 0$ and by Schwarz's Lemma, $|a(z)| \le 1/|z|$ for all $z \in \Delta$. Given $R > 1$, then $f \in \mathscr{F}_{p_0}$ implies that f is K_R-quasiconformal on $|z| > R$ where $K_R = (R+1)/(R-1)$. In particular, since $|a(p_0)| = |\bar{\gamma}/(1+\gamma)| \le 1/p_0$, it follows that

$$\left|\gamma - \frac{1}{p_0^2 - 1}\right| \le \frac{p_0}{p_0^2 - 1}. \tag{12}$$

In other words, the set of possible γ is bounded and stays away from the vertical line $\operatorname{Re}\gamma = -1/2$, and therefore, it is a compact set of the domain $\{\gamma;\, \operatorname{Re}\gamma > -1/2\}$. It follows then that $\mathscr{F}_{p_0}$ forms a normal family on $\{z;\, r \le |z| \le R\}$ for all $1 < r < R < \infty$. Applying the transformation $F(s) = 1/f(1/s)$, $|s| < 1$, we conclude that $\mathscr{F}_{p_0}$ is a normal family on $|z| > p_0$, which implies that $\mathscr{F}_{p_0}$ is a normal family on $|z| > r$ for all $r > 1$. Finally, applying Cantor's Diagonal procedure to the restrictions of $\mathscr{F}_{p_0}$ to $|z| > 1 + 1/n$, $n \in \mathscr{N}$, we deduce that $\mathscr{F}_{p_0}$ is itself a normal family. The compactness follows now by standard arguments for quasiconformal mappings. ∎

One may be asking why we do not just change the normalization for the definition of Σ_{Lh} (Σ'_{Lh} resp.) by requiring $|h(\infty)| = -p^{1+2\operatorname{Re}\gamma}$ instead of $|h(\infty)| = 1$. The answer to this question is given in the next result.

THEOREM 3.2. *Let $f \in \Sigma_{Lh}$. Then there is a point e^{it_0} of the unit circle ∂U which belongs to $\partial f(\Delta)$.*

Proof. Let $\omega(\zeta)$ be defined by the relations (7) and (8), i.e., put

$$\Psi(z) = \frac{f(z)}{z|z|^{2\beta}((z-p)/(1-\bar{p}z))|(z-p)/(1-\bar{p}z)|^{2\gamma}},$$

$$z \in \Delta \text{ and } \omega(\zeta) = \Psi\left(\frac{1}{\zeta}\right),\ \zeta \in U.$$

Then $\omega(\zeta) = H(\zeta)\overline{G(\zeta)}$ and $|\omega(0)| = |h(0)| = 1$. Since H and G are analytic and nonvanishing on U, we have by the maximum and the minimum modulus principle

$$\varliminf_{|z|\to 1} |f(z)| \leq 1 \leq \varlimsup_{|z|\to 1} |f(z)|. \tag{13}$$

Since $f(\Delta)$ and the omitted set $\mathbb{C} \setminus f(\Delta)$ are connected sets, there exists at least one $e^{it_0} \in \partial U$ such that $e^{it_0} \in \partial f(\Delta)$. ∎

4. ONE POINTED UNIVALENT LOGHARMONIC MAPPINGS

Let f be a univalent logharmonic mapping defined on Δ, $f(\infty) = \infty$. Then there is a real number α and a positive constant A such that $Af(e^{i\alpha}z)$ belongs to Σ_{Lh}. By Theorem 3.2, there is a point e^{it_0} of ∂U which belongs to the complement of $Af(\Delta)$. Therefore, if f does not vanish on Δ, then the set of omitted values is a continuum. In other words, there is no univalent logharmonic mapping f defined on Δ such that $f(\infty) = \infty$ and $f(\Delta) = \mathbb{C} \setminus \{0\}$. We shall see that 0 is an exceptional point, since for each $w_0 \in \mathbb{C} \setminus \{0\}$, there are univalent logharmonic mappings f such that $f(\Delta) = \mathbb{C} \setminus \{w_0\}$. In this section we shall characterize these mappings. Again, we assume that $p > 1$. Let $f \in \Sigma_{Lh}$ and let w_0 be an omitted value of f. Applying a rotation to the image $f(\Delta)$ we may assume that $w_0 = 1$ and we restrict ourselves to the subclass

$$\Sigma^*_{Lh} = \{f \in \Sigma_{Lh}, p > 1, w_0 = 1 \notin f(\Delta)\}. \tag{14}$$

Suppose now that $f \in \Sigma^*_{Lh}$ and that $f(e^{it}) \equiv 1$ on an interval $\mathscr{J} = \{e^{it}, \alpha < t < \beta\}$ of the unit circle ∂U. Then $|a| = 1$ on $\mathscr{J}$ and a admits an analytic continuation across $\mathscr{J}$. Since $a(1/\bar{z}) = 1/\overline{a(z)}$, it follows that $|a| > 1$ for $z \in U$ close to $\mathscr{J}$. Indeed, a' does not vanish on $\mathscr{J}$ since $|a| < 1$ on Δ. In our first lemma, we shall see that f admits an extension of the form $f(1/\bar{z}) = 1/f(z)$ which we shall call an anti-logharmonic extension of f (since $|a| > 1$).

LEMMA 4.1. *Let $f \in \Sigma^*_{Lh}$ and suppose that the boundary values $f(e^{it})$ exist and satisfy $f(e^{it}) \equiv 1$ on some interval $\mathscr{J} = \{e^{it}, \alpha < t < \beta, 0 < \beta - \alpha < 2\pi\}$ of ∂U. Then f admits an anti-logharmonic extension across $\mathscr{J}$ and we have $f(1/\bar{z}) = 1/f(z)$. Furthermore, h and g admit an analytic extension across $\mathscr{J}$ and we have*

$$\overline{h\left(\frac{1}{\bar{z}}\right)} = \frac{z(z-p)}{(1-pz)g(z)} \qquad \textit{and} \qquad \overline{g\left(\frac{1}{\bar{z}}\right)} = \frac{(1-pz)}{z(z-p)h(z)}$$

for all $z \in \Omega = \{z;\ 1/p < |z| < p$ and $\alpha < \arg z < \beta\}$.

Proof. Define $D_0 = \{z;\ 1 < |z| < p \text{ and } \alpha < \arg z < \beta\}$. Then

$$\begin{aligned} F(z) &= \log f(z) = H(z) + \overline{G(z)} \\ &= \left[(1+\beta)\log z + (1+\gamma)\log\frac{z-p}{1-pz} + \log h(z)\right] + \beta\log\bar{z} \\ &\qquad + \gamma\log\frac{\bar{z}-p}{1-p\bar{z}} + \log\overline{g(z)} \end{aligned} \tag{15}$$

can be defined as a (univalent) harmonic mapping on D_0. By the reflection principle for harmonic functions, it follows that F admits a harmonic extension across $\mathscr{J}$ and we have

$$F\left(\frac{1}{\bar{z}}\right) = -F(z), \qquad z \in \Omega = D_0 \cup \mathscr{J} \cup D_1, \tag{16}$$

where $D_1 = \{z;\ 1/\bar{z} \in D_0\}$. Since Ω is a simply connected domain and since both functions $\operatorname{Re}\log h\bar{g} = \ln|hg|$ and $\operatorname{Im}\log h\bar{g} = \arg(h/g)$ can be extended to harmonic functions on Ω we conclude that hg and h/g are analytic functions on Ω which implies that h and g are analytic on Ω. Next, we substitute (15) in (16) and we get for $z \in \Omega$,

$$\begin{aligned} &-(1+\beta)\log\bar{z} - (1+\gamma)\log\frac{\bar{z}-p}{1-p\bar{z}} + \log h\left(\frac{1}{\bar{z}}\right) \\ &\qquad - \beta\log z - \gamma\log\frac{z-p}{1-pz} + \log \overline{g\left(\frac{1}{\bar{z}}\right)} \\ &\qquad = -(1+\beta)\log z - (1+\gamma)\log\frac{z-p}{1-pz} - \log h(z) \\ &\qquad\qquad - \beta\log\bar{z} - \gamma\log\frac{\bar{z}-p}{1-p\bar{z}} - \log\overline{g(z)} \end{aligned}$$

or

$$\begin{aligned} &\log\bar{z} + \log\frac{\bar{z}-p}{1-p\bar{z}} - \log\overline{g(z)} - \log h\left(\frac{1}{\bar{z}}\right) \\ &\qquad = \log z + \log\frac{z-p}{1-pz} + \log h(z) + \log \overline{g\left(\frac{1}{\bar{z}}\right)}. \end{aligned} \tag{17}$$

Since the right-hand side is an analytic function on Ω and the left-hand side is anti-analytic in Ω, we conclude that

$$\log z + \log\frac{z-p}{1-pz} + \log h(z) + \log \overline{g\left(\frac{1}{\bar{z}}\right)} \equiv \text{const. on } \Omega \tag{18}$$

or, equivalently,

$$\overline{g\left(\frac{1}{\bar{z}}\right)} = A\frac{1-pz}{z(z-p)h(z)}, \qquad A \neq 0, \text{ for all } z \in \Omega. \tag{19}$$

For $z = e^{it} \in \mathscr{J}$, we get

$$f(z) = \frac{z(z-p)}{(1-pz)}h(z)\overline{g(z)} = \frac{z(z-p)}{(1-pz)}h(z)\overline{g\left(\frac{1}{\bar{z}}\right)} = A = 1.$$

Therefore, we have for all $z \in \Omega$:

$$\overline{h\left(\frac{1}{\bar{z}}\right)} = \frac{z(z-p)}{(1-pz)g(z)} \qquad \text{and} \qquad \overline{g\left(\frac{1}{\bar{z}}\right)} = \frac{(1-pz)}{z(z-p)h(z)}. \tag{20}$$

Finally, (16) and $f = e^F$ imply $f(1/\bar{z}) = 1/f(z)$. ∎

COROLLARY 4.2. *Let $f \in \Sigma^*_{Lh}$ satisfy $f(e^{it}) \equiv 1$ on ∂U. Then f admits an anti-logharmonic extension across ∂U onto the domain $U \setminus \{1/p\}$ and f satisfies $f(1/\bar{z}) = 1/f(z)$ $\forall z \in \overline{\mathbb{C}} \setminus \{1/p\}$. Furthermore, h and g can be extended to analytic functions on $\overline{\mathbb{C}} \setminus [\{1/p\} \cup \{0\}]$. We have for all $z \in \Omega$:*

$$g(z) = \frac{e^{i\alpha}}{h(z)}, \tag{21}$$

$$\overline{h\left(\frac{1}{\bar{z}}\right)} = \frac{z(z-p)}{(1-pz)g(z)} \tag{22}$$

and

$$\overline{g\left(\frac{1}{\bar{z}}\right)} = \frac{(1-pz)}{z(z-p)h(z)}. \tag{23}$$

Proof. Applying Lemma 4.1 to the interval $\mathscr{J}_1 = \{e^{it};\ 0 \leq |t| < 3\pi/4\}$ and $\mathscr{J}_2 = \{e^{it};\ \pi/4 < |t| \leq \pi\}$ we see that f admits an anti-logharmonic extension across ∂U onto the domain $U \setminus \{1/p\}$. Furthermore, (22) and (23) hold and $f(1/\bar{z}) = 1/f(z)$ for all $z \in \overline{\mathbb{C}} \setminus \{1/p\}$. Next, $\chi(\zeta) = h(1/\zeta)g(1/\zeta)$ is a nonvanishing analytic function defined on the unit disk U whose boundary values $\chi(e^{it})$ exist on ∂U and we have $|\chi(e^{it})| \equiv 1$. By the maximum and minimum modulus principle we conclude that $\chi(\zeta) \equiv e^{i\alpha}$ on U. Therefore, we have $g(z) = e^{i\alpha}/h(z)$ for all $z \in \Delta$, and hence, also for all $z \in \overline{\mathbb{C}} \setminus \{1/p\}$. ∎

The next lemma gives a necessary and sufficient condition for a logharmonic mapping f to belong to Σ^*_{Lh} and to satisfy $f(\Delta) = \mathbb{C} \setminus \{1\}$.

LEMMA 4.3. *Let f be a logharmonic mapping on Δ. Then $f \in \Sigma_{Lh}^*$ and $f(\Delta) = \mathbb{C} \setminus \{1\}$ if and only if f is of the form*

$$f(z) = \bar{z}\frac{(z-p)}{(1-p\bar{z})}|z|^{2\beta}\left|\frac{z-p}{1-pz}\right|^{2\gamma}, \qquad p > 1. \tag{24}$$

Remark. There are mappings of the form (24) which are not logharmonic (see, for example, the mapping defined in (31)).

Proof. Suppose that $f \in \Sigma_{Lh}^*$ and that $f(\Delta) = \mathbb{C} \setminus \{1\}$. Since f is logharmonic on Δ, f admits the representation

$$f(z) = z|z|^{2\beta}\left(\frac{z-p}{1-pz}\right)\left|\frac{z-p}{1-pz}\right|^{2\gamma} h(z)\overline{g(z)}, \tag{25}$$

where $p > 1$, $\operatorname{Re}\beta > -1/2$, $\operatorname{Re}\gamma > -1/2$, and where h and g are nonvanishing analytic functions on $\Delta \cup \{\infty\}$. By Corollary 4.2, we have $g(z) = e^{i\alpha}/h(z)$ and therefore, (25) becomes

$$f(z) = e^{i\alpha}z|z|^{2\beta}\left(\frac{z-p}{1-pz}\right)\left|\frac{z-p}{1-pz}\right|^{2\gamma}\frac{h(z)}{\overline{h(z)}}. \tag{26}$$

Combining (21), (22), and (23), we get for all $z \in \overline{\mathbb{C}}$

$$\overline{h(1/\bar{z})} = \frac{e^{-i\alpha}z(z-p)h(z)}{(1-pz)} \tag{27}$$

which shows that h has a simple zero at $z = 1/p$ and a simple pole at $z = 0$. Therefore, we have

$$h(z) = e^{i\alpha}\frac{pz-1}{pz} \tag{28}$$

and, by (23),

$$g(z) = \frac{pz}{pz-1}.$$

Since $f(1) = h(1)\overline{g(1)} = 1$, we get $e^{i\alpha} = 1$. From the substitution of (28) into (26), we conclude that

$$\begin{aligned} f(z) &= z|z|^{2\beta}\left(\frac{z-p}{1-pz}\right)\left|\frac{z-p}{1-pz}\right|^{2\gamma}\frac{pz-1}{pz}\,\frac{p\bar{z}}{p\bar{z}-1} \\ &= \bar{z}|z|^{2\beta}\left(\frac{z-p}{1-p\bar{z}}\right)\left|\frac{z-p}{1-pz}\right|^{2\gamma} \end{aligned}$$

which is of the form (24). Conversely, let f be a logharmonic mapping of the form (24). Then

$$f(e^{it}) = e^{-it}\frac{(e^{it} - p)}{(1 - pe^{-it})} \equiv 1$$

and the univalence of f follows from Corollary 2.4 and the definition of a logharmonic mapping. ∎

As we have already remarked, not all mappings of the form (24) belong to Σ^*_{Lh} even if $\operatorname{Re}\beta > -1/2$ and $\operatorname{Re}\gamma > -1/2$. We had to require that f is logharmonic, i.e., $|a(z)| < 1$ on Δ. Direct calculations show that the second dilatation function is

$$a(z) = \frac{\overline{\beta}pz^2 - \left[1 + \overline{\gamma}(1 - p^2) + \overline{\beta}(1 + p^2)\right]z + (1 + \overline{\beta})p}{\beta p - \left[1 + \gamma(1 - p^2) + \beta(1 + p^2)\right]z + (1 + \beta)pz^2} \tag{29}$$

which is of the form

$$a(z) = e^{i\delta}\frac{(z - A)(z - B)}{(1 - \overline{A}z)(1 - \overline{B}z)}, \qquad \text{if } \beta \neq 0 \tag{30}$$

and

$$a(z) = \frac{e^{i\delta}}{z}\,\frac{z - C}{1 - \overline{C}z} \qquad \text{or} \qquad a(z) = \frac{1}{z^2}, \qquad \text{if } \beta = 0.$$

The condition $|a| < 1$ is equivalent to the requirement $|A| > 1$ and $|B| \geq 1$, if $\beta \neq 0$ and $|C| > 1$, if $\beta = 0$. For instance,

$$f(z) = \bar{z}|z|^2\frac{z - 4}{1 - 4\bar{z}} \tag{31}$$

is not logharmonic since $a(z) = ((1 - 2z)(z - 4))/((1 - 4z)(z - 2))$ and $a(2) = \infty$.

Our main result of this section is the following theorem.

THEOREM 4.4. *A mapping f belongs to Σ^*_{Lh} and $f(\Delta) = \mathbb{C} \setminus \{1\}$ if and only if f is of the form* (24),

$$f(z) = \bar{z}|z|^{2\beta}\frac{(z - p)}{(1 - p\bar{z})}\left|\frac{z - p}{1 - pz}\right|^{2\gamma}, \qquad p > 1,$$

where β and γ satisfy the inequality

$$\operatorname{Re}\gamma > -1/2 \qquad \textit{and}$$

$$\left|\frac{\beta(1 + \overline{\gamma}) - \gamma(1 + \overline{\beta})}{1 + \gamma + \overline{\gamma}} - \frac{1}{p^2 - 1}\right| \leq \frac{p}{p^2 - 1}. \tag{32}$$

Proof. First, assume that $\gamma = 0$. Then $a(z)$ vanishes at the points $z = p$ and $z = (1 + \overline{\beta})/\overline{\beta}p$, and the condition $|a| < 1$ on Δ is satisfied if and only if $|1 + \beta| \geq |\beta| p$. Next, assume that $\gamma \neq 0$. Since $a(p) = \overline{\gamma}/(1 + \gamma)$, it follows that $\operatorname{Re}\gamma > -1/2$. Apply the transformation $\hat{f} = Tf = f|f|^{2\tau}$, where $\tau = -\gamma/(1 + 2\operatorname{Re}\gamma)$. Then $\hat{f} \in \Sigma_{Lh}^{*}$ if and only if $f \in \Sigma_{Lh}^{*}$ and we have $\hat{\beta} = \beta + 2\tau \operatorname{Re}\beta + \tau$. Furthermore, $\hat{f}$ is a solution of (4) with respect to $\hat{a} = ((1 + \gamma)a - \overline{\gamma})/((1 + \overline{\gamma}) - \gamma a)$. In other words, f is logharmonic if and only if $\hat{f}$ is logharmonic. Hence, f is logharmonic if and only if $|1 + \hat{\beta}| \geq |\hat{\beta}| p$ or equivalently,

$$\left| \hat{\beta} - \frac{1}{p^2 - 1} \right| \leq \frac{p}{p^2 - 1}.$$

Therefore, (32) is a necessary and sufficient condition for (24) to define a logharmonic mapping. ∎

For $\beta \neq 0$, we have seen that the dilatation function a of the mappings $f \in \Sigma_{Lh}^{*}$, $f(\Delta) = \mathbb{C} \setminus \{1\}$, is of the form (30) where $e^{i\delta} = \overline{\beta}/\beta$, $|A| > 1$ and $|B| \geq 1$. It is natural to ask if for each such a there is a one pointed mapping $f \in \Sigma_{Lh}^{*}$ which satisfies the partial differential equation (4). From (29) we conclude that $ABe^{i\delta} = (1 + \overline{\beta})/\beta$ and $AB = (1 + \overline{\beta})/\overline{\beta}$. Hence $e^{i\delta}A$ and B cannot be chosen independently and the answer to the above question is negative.

Our next result is a nonexistence theorem.

THEOREM 4.5. *Let $\hat{f}$ belong to Σ_{Lh}^{*} so that $\hat{f}(\Delta) = \mathbb{C} \setminus \{1\}$ and let $\hat{a}$ be its second dilatation function. Then there is no other univalent solution f of* (4) *with respect to $\hat{a}$ which belongs to Σ_{Lh}^{*} and satisfies $f(p) = \hat{f}(p) = 0$.*

Proof. Let $f \in \Sigma_{Lh}^{*}$ be a solution of (4) with respect to $\hat{a}$. Define $\Phi(\zeta) = f(1/\zeta)/\hat{f}(1/\zeta)$. Then Φ is a bounded logharmonic mapping defined on the unit disk U which is a solution of (4) with respect to the dilatation function $a_1(\zeta) = \hat{a}(1/\zeta)$. Since $\hat{f}(e^{it}) \equiv 1$, the cluster sets $\{\Phi(e^{it})\}$, $0 \leq t < 2\pi$, coincide with the cluster sets $\{f(e^{it})\}$, $0 \leq t < 2\pi$. Hence, either the interior of the omitted set $\mathbb{C} \setminus f(\Delta)$ is empty or a nonempty domain of $\mathbb{C}$. In the first case, we conclude that Φ is a constant and, therefore, we get $f = c\hat{f}$ for some $c \in \mathbb{C}$. Moreover, there is an $e^{it_0} \in \partial U$ such that the cluster set of f at e^{it_0} contains the point 1. Hence, $c = 1$. Now, suppose that Φ is not a constant. Then Φ is a bounded and open mapping. Hence, $\Phi(U)$ is a nonempty domain G of $\mathbb{C}$. Since the cluster set of Φ at e^{it} coincides with the cluster set of f at e^{-it}. They form a bounded connected set T of $\mathbb{C}$, it follows that G is a simply connected domain such that $\partial G = T$. Choose a $w_0 \in G$ and suppose that $\{z = e^{it};\ 0 \leq t \leq 2\pi\}$ winds once around the unit circle ∂U in the positive direction. Then the total change of $\arg(f - w_0)$ is 2π and, therefore, the total change of $\arg(\Phi - w_0)$ is -2π. But this is a contradiction to the fact that Φ is an open orientation-preserving mapping on U. Hence, Φ is a constant and Theorem 4.5 is proved. ∎

5. PROOF OF THE MAIN RESULT

Recall that a domain Ω is harmonically accessible from a domain D if there exists a univalent harmonic mapping from D onto Ω. Let K be a compact set of $\mathbb{C}$ which satisfies the following two properties:

(5.1) If $z \in K$ then $z + 2\pi i n \notin K$ $\forall n \in \mathbf{Z} \setminus \{0\}$.

(5.2) K and $\mathbb{C} \setminus K$ are connected sets.

Define

$$K_n = \{z = \zeta + 2\pi i n;\ \zeta \in K\}, \qquad n \in \mathscr{Z},$$

$$D^* = \mathbb{C} \setminus \bigcup_{n \in \mathscr{Z}} K_n$$

and

$$\Omega^* = \mathbb{C} \setminus \bigcup_{n \in \mathscr{Z}} \{2\pi i n\}.$$

We shall show that Ω^* is harmonically accessible from D^*. In other words, we have the following result:

THEOREM 5.1. *There exists a univalent harmonic and orientation-preserving mapping F from D^* onto Ω^* such that $F(\infty) = \infty$.*

Remark. The mapping F is not uniquely determined.

Proof. If K is a single point q of $\mathbb{C}$, consider the mapping $w = z - q$ which shows that D^* and Ω^* are conformally equivalent and, hence, also harmonically equivalent. Suppose now that K is a continuum. Define $s = f_1(z) = e^z$. Then we have $f_1(D^*) = \mathbb{C} \setminus (E \cup \{0\})$ where $E = f_1(K) = f_1(K_n)$ is a compact set which does not contain the origin for all $n \in \mathscr{Z}$. Furthermore, E and $\mathbb{C} \setminus E$ are connected sets. Define

$$D_1^* = f_1(D^*) \cup \{0\}$$

and

$$t = f_2(s) = \overline{\phi(s)}\left(\frac{\phi(s) - \phi(0)}{1 - \phi(0)\overline{\phi(s)}}\right),$$

where ϕ is the conformal mapping from D_1^* onto $\Delta = \mathbb{C} \setminus \overline{U}$, $\phi(\infty) = \infty$, and $\phi(0) > 1$. Since (32) is satisfied for $\gamma = \beta = 0$, f_2 is a univalent logharmonic mapping from D_1^* onto $\mathbb{C} \setminus \{1\}$ satisfying $f_2(\infty) = \infty$ and $f_2(0) = 0$. Finally, put $F(z) = \log f_2 \circ f_1(z)$. We show that F has the desired properties.

Fix $z_0 \in D^*$. Then the possible values of $F(z_0)$ are $w_0 + 2\pi i m$, $m \in \mathscr{Z}$, for some fixed $w_0 \in \Omega^*$. The same holds for all the possible values of $F(z_0 + 2\pi i k)$, $k \in \mathscr{Z}$. Let $\gamma \equiv z(t)$, $0 \le t \le 1$, be a closed curve in D^* which starts and ends at z_0. Suppose that $F(z(0)) = w_0$. Then $F(z(1)) = w_0 + 2\pi i n$ for some $n \in \mathscr{Z}$. It follows that $f_2 \circ f_1(z(t))$, $0 \le t \le 1$, is a continuous closed curve winding n times (in the positive sense) around the origin. The same holds for the continuous closed curve $f_1(z(t))$, $0 \le t \le 1$, from which we conclude that $z(1) = z_0 + 2\pi i n = z_0$ and, therefore, we have $n = 0$. In other words, F is a well-defined function on D^*. Applying the same argument backwards for a closed curve in Ω^* starting and ending at w_0 we conclude that F is univalent on D^*. Finally, for each $z_0 \in D^*$, there is a neighborhood $V(z_0)$ belonging to D^* where F is of the form

$$F(z) = H(z) + \overline{G(z)},$$

which shows that F is harmonic on $V(z_0)$. Since $f_2(s)$ is an orientation-preserving mapping defined on D_1^*, it follows that F is a univalent harmonic and orientation-preserving mapping from D^* onto Ω^* and Theorem 5.1 is established. ∎

We finish this article with an elementary application to nonparametric minimal surfaces over the domain $\Omega^* = \mathbb{C} \setminus \bigcup_{n \in \mathscr{Z}} \{2\pi i n\}$. Let $\mathbf{S} = (u, v, s(u, v))$ be a nonparametric minimal surface over the domain Ω^*, which is represented by a univalent harmonic and orientation-preserving mapping F from D^* onto Ω^* by

$$u(z) = \operatorname{Re} F(z)$$

$$v(z) = \operatorname{Im} F(z)$$

$$s(z) = \pm 2\operatorname{Im} \int \sqrt{-\overline{F_{\bar z}} F_z}\, dz.$$

Moreover we may assume that F is a solution of the elliptic partial differential equation $\overline{F_{\bar z}} = A F_z$, where $\sqrt{A} \in H(D^*)$ and $|A| < 1$ on D^*. Since the Riemannian metric of $\mathbf{S}$ is $ds^2 = |F_z|^2(1 + |A|)^2 |dz|^2$, it follows that $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are isothermal parameters for $\mathbf{S}$. It is interesting to note that the exterior unit normal vector $\vec{\mathbf{n}}(z) = (\mathbf{n}_1(z), \mathbf{n}_2(z), \mathbf{n}_3(z))$, $\mathbf{n}_3(z) \ge 0$, to the minimal surface $\mathbf{S}$ (known as the Gauss mapping) depends only on the second dilatation function $A(z)$ of F. More precisely, we have

$$\vec{\mathbf{n}} = \left(2\operatorname{Im}\sqrt{A}, 2\operatorname{Re}\sqrt{A}, (1-|A|)\right)/(1+|A|). \tag{33}$$

The inverse of the stereographic projection of the Gauss mapping $\vec{\mathbf{n}}$, $i/\sqrt{A(z)}$, is called the Weierstrass parameter (see, e.g., [12]).

Suppose now, in addition, that F satisfies $F(z+2\pi i) = F(z) + 2\pi i$. It follows then that the Gauss map $\vec{\mathbf{n}}(z)$, defined in (31), is periodic, i.e., that $\vec{\mathbf{n}}(z+2\pi i) = \vec{\mathbf{n}}(z)$.

(a) *K is a single point $q \in \mathbb{C}$.* Then F admits a univalent harmonic extension across the points $q + 2\pi i n$, $n \in \mathscr{Z}$, onto the whole plane $\mathbb{C}$. On the other hand, $\sqrt{A}$ admits an analytic extension onto $\mathbb{C}$ satisfying $|A| < 1$ on $\mathbb{C}$. Therefore, A is a constant and F is an affine transformation. In other words, **S** is a plane.

(b) *K is a continuum.* Let f_1, ϕ, and f_2 be defined as at the beginning of this section. Then $f_2 \circ \phi^{-1}$ is a univalent logharmonic mapping from Δ onto the pointed plane $\mathbb{C} \setminus \{1\}$ normalized by $f_2 \circ \phi^{-1}(\infty) = \infty$ and $f_2 \circ \phi^{-1}(p) = 0$ for some $p = \phi(0) > 0$. The second dilatation function $a(z)$ defined in (29) is connected with $A(z)$ by the relation

$$A(z) = a(\phi(e^z)), \qquad z \in D^*.$$

Since $\sqrt{A(z)} = \sqrt{a(\phi(e^z))} \in H(D^*)$, each zero of A has to be of even order. Hence, together with (30),

$$A(z) = e^{i\delta}\left[\frac{\phi(e^z) - B}{1 - \bar{B}\phi(e^z)}\right]^2 \qquad \text{or} \qquad A(z) = \frac{1}{(\phi(e^z))^2}$$

for some $\delta \in \mathscr{R}$ and some B, $|B| > 1$. Then the following two cases can appear:

(I) **S** has no horizontal tangent plane and for all $w = u + iv$, v fixed, we have either $\lim_{u\to\infty} \vec{\mathbf{n}}(w) = (0,0,1)$ or $\lim_{u\to-\infty} \vec{\mathbf{n}}(w) = (0,0,1)$.

(II) **S** admits exactly one horizontal tangent plane over each strip of width 2π, $\{w = u + iv;\ u \in \mathscr{R}$ and $v_0 \le v \le v + 2\pi\}$.

REFERENCES

1. Z. Abdulhadi and D. Bshouty, Univalent mappings in $H.\bar{H}$, *Trans. Amer. Math. Soc.* **305** (1988), 841–849.
2. Z. Abdulhadi and W. Hengartner, Spirallike logharmonic mappings, *Complex Variables Theory Appl.* **9** (1987), 121–130.

3. Z. Abdulhadi and W. Hengartner, Univalent harmonic mappings on the left-half plane with periodic dilatation, *in* ‘‘Univalent Functions, Fractional Calculus, and Their Applications,’’ pp. 13–28, Ellis Horwood, 1989.
4. D. Bshouty and W. Hengartner, Univalent solutions of the Dirichlet problem for ring domains, *Complex Variables Theory Appl*. **21** (1993), 159–169.
5. J. Eelle and L. Lemaire, A report on harmonic maps, *Bull. London. Math. Soc*. **10** (1978), 1–68.
6. J. Eelle and J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math*. **86** (1964), 109–160.
7. W. Hengartner and G. Schober, Univalent harmonic mappings, *Trans. Amer. Math. Soc*. **299**, No. 1 (1987), 1–31.
8. W. Hengartner and G. Schober, Univalent harmonic exterior and ring mappings, *J. Math. Anal. Appl*. **156** (1991), 154–171.
9. J. Jost, ‘‘Harmonic Maps between Surfaces,’’ Lecture Notes in Mathematics, Springer-Verlag, New York, 1984.
10. J. Jost, ‘‘Two-Dimensional Geometric Variational Problems,’’ Wiley, Toronto, 1991.
11. J. C. C. Nitsche, ‘‘Lectures on Minimal Surfaces,’’ Vol. 1, Springer-Verlag, New York, 1989.
12. R. Osserman, ‘‘A Survey of Minimal Surfaces,’’ Dover, New York, 1986.
13. R. Schoen, The role of harmonic mappings in rigidity and deformation problems, *in* ‘‘Lecture Notes in Pure and Appl. Math. No. 143,’’ pp. 179–200, Dekker, New York, 1993.
14. R. Schoen, The theory and applications of harmonic mappings between Riemannian manifolds, preprint.